PERFECT POPULATION TRANSFER
IN PULSE-DRIVEN QUANTUM CHAINS

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The quantum dynamics of a pulse-driven finite-dimensional quantum chain with only nearest-neighbor coupling is studied. We extend the concept of Rabi oscillations from two-level quantum systems to the multi-level quantum chains. The time-dependent quantum dynamics and solutions producing perfect population transfer are obtained for up to five-level quantum chains. The Gröbner basis analysis technique is used to generalize the results and get all analytical solutions for perfect population transfer. Explicit formulas for the solutions up to nine levels are presented. These results could be used to design control strategies for general finite-dimensional quantum systems.

Keywords: Quantum control; population transfer; quantum chains.

1. Introduction

Atoms and molecules prepared in specific quantum states are crucial in many areas of modern atomic and molecular physics, such as quantum information, atom optics, laser-controlled chemical reactions, and state-to-state collision. The basic idea behind state preparation is to begin with an atom or molecule in a specified discrete quantum state and then, by exposing the system to controlled pulses of radiation, force it into a desired target state. Various scenarios exist, including optimal control, coherent control, stimulated Raman adiabatic passage (STIRAP) control, etc. Increasing numbers of control experiments, including on complex systems, employ closed-loop optimal control. Quantum measurements could also

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be tools to manipulate population transfer.\textsuperscript{9,10} A recent paper reviews coherent manipulations of atoms using lasers.\textsuperscript{11}

Analytical solutions for population transfer have been dealt with in many papers. Some special analytical solutions for \(N\)-level ladder-like structure chains have been discussed.\textsuperscript{12,13} Specific pulses and phase modulations were presented in analytic form to produce particular degrees of inversion of multi-level atoms.\textsuperscript{14} The dynamics of an \(N\)-level system with a specific “Raman” structure of levels interacting in a lossless cavity with an \((N-1)\)-mode resonant quantized field was studied.\textsuperscript{15} The case of degenerate multi-state systems excited by a coherent pulse has been considered.\textsuperscript{16} In this paper, we present analytical solutions for perfect population transfer in \(N\)-level chains. Quantum dissipation is not considered, because the driving pulses are assumed to be short. When dissipation is important, its effect in finite-dimensional systems has been considered by Xu et al.\textsuperscript{17,18} Recent studies\textsuperscript{19,20} investigated the effects of field noise and decoherence, and these works show that controlled quantum dynamics can survive intense field noise and decoherence with the prospect of even cooperating with them under special circumstances. The foundation of cooperating between the control field and noise or decoherence was explained by a perturbation theory analysis.\textsuperscript{20,21} This paper goes beyond the perturbation approximation and deals with control fields of moderate strength within the rotating wave approximation (RWA).\textsuperscript{22} The resultant dynamics of the quantum systems and the solutions for perfect population transfer are studied here.

Section 2 presents the control model for population transfer in a pulse-driven quantum chain. In Sec. 3, we extend the concept of Rabi oscillations to multi-level chains and attain the solutions for pulses producing exact population transfer in up to five-level chains. Section 4 utilizes a Gröbner basis analysis to extend this work and obtains analytical solutions for control pulses that induce perfect population transfer in \(N\)-level chains. The optimal solutions with minimal fluence of Rabi frequencies are also presented. Finally, some conclusions and discussion are given in Sec. 5.

2. Model of Pulse-Driven Quantum Chains

The model presented here is similar to the models that were used to explain the cooperative effect between the control field and noise\textsuperscript{20} or dissipation.\textsuperscript{21} We consider the excitation along a chain of nondegenerate transitions and energy levels, each linked only to its nearest neighbors. One can think of this system as a nonlinear oscillator\textsuperscript{23–25} or a spin with \(S > 1\), with nonequidistant energy levels. The system dynamics is governed by the Hamiltonian \(H\),

\[
H = H_0 - \mu E(t), \quad (1a)
\]

\[
H_0 = \sum_{n=1}^{N} \varepsilon_n |n\rangle\langle n|, \quad (1b)
\]
where $|n\rangle$ is an eigenstate of $H_0$ with the associate energy $\varepsilon_n$ in the absence of radiation and $\mu$ is the dipole operator whose elements take the form

$$\mu_{nn'} = \mu_n \delta_{n', n+1} + \mu_{n'} \delta_{n, n'+1}. \quad (2)$$

The $N$-level system consists of an initially occupied ground state $|1\rangle$, $N - 2$ intermediate states $|n\rangle$, $n = 2, 3, \ldots, N - 1$, and a final target state $|N\rangle$ we want to reach at final time $T_f$. These states are coupled with an external laser pulse $E(t)$,

$$E(t) = 2s(t) \sum_{n=1}^{N-1} A_n \cos(\omega_n t + \theta_n) = s(t) \sum_{n=1}^{N-1} A_n e^{i(\omega_n t + \theta_n)} + c.c., \quad (3)$$

where $\omega_n$ are the frequencies of the radiation and $s(t)$ is the pulse envelope function. The controls are the amplitudes $A_n$ and phases $\theta_n$. Figure 1 depicts the quantum chain driven by the laser pulse. We seek a wave function of the form

$$|\psi(t)\rangle = \sum_{n=1}^{N} C_n(t) e^{-i(\varepsilon_n t + \Theta_n)} |n\rangle, \quad (4a)$$

$$\Theta_n = \theta_1 + \theta_2 + \cdots + \theta_{n-1}, \quad (4b)$$

where $C_n(t)$ are complex probability amplitudes. The initial condition at $t = 0$ specifies that $C_1 = 1$ and $C_n = 0$ for $1 < n \leq N$. We want to find $C_N$ at $t = T_f$.

If each component of the pulse is only exactly resonant with one transition of the

![Fig. 1. The model of the finite-dimensional quantum system driven by a laser pulse. Here $\omega_k$, $k = 1, \ldots, N - 1$ are the frequencies of the radiation and $C_k(t), k = 1, \ldots, N$ are complex probability amplitudes of the wave function.](image)
system,

\[ \omega_k = \varepsilon_{k+1} - \varepsilon_k, \quad (5) \]

and the field is not very strong, we can use the RWA to omit all nonresonant terms. Therefore, the Schrödinger equation becomes

\[ i \frac{d}{dt} C_n(t) = -s(t) [\mu_{n-1} A_{n-1} C_{n-1}(t) + \mu_n A_n C_{n+1}(t)], \quad (6) \]

or in matrix form

\[ \frac{d}{dt} \vec{C}(t) = i s(t) H \vec{C}. \quad (7) \]

Here, we set \( \hbar = 1 \) and the transformed Hamiltonian \( H \) is a tridiagonal matrix,

\[
H = \begin{bmatrix}
0 & \Omega_1 & \cdots & 0 & 0 \\
\Omega_1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \Omega_{N-1} \\
0 & 0 & \cdots & \Omega_{N-1} & 0
\end{bmatrix},
\]

which only depends on the Rabi frequencies \( \Omega_k \),

\[ \Omega_k = \mu_k A_k. \quad (9) \]

All \( \Omega_k \) are positive, which can be assured by appropriately defining the phase in Eq. (4b). Information transfer over spin chains can be also described by a similar model.\(^{26}\)

It is easy to verify that the solution of Eq. (7) is

\[ \vec{C}(t) = \exp(iH\tau_e(t)) \vec{C}(0), \quad (10) \]

with effective pulse time \( \tau_e(t) \) defined as

\[ \tau_e(t) = \int_0^t s(t') dt'. \quad (11) \]

Hence, the outcome of the control field is

\[
O[E(t)] = |C_N(T_f)|^2, \quad (12a)
\]

\[
C_N(T_f) = (e^{iHT_f})_{N,1}, \quad (12b)
\]

with the effective pulse duration being the integration of the pulse envelope function,

\[ T_e = \int_0^{T_f} s(t) dt. \quad (13) \]
3. Rabi Oscillations and Perfect Population Transfer

Solutions have already been found for the Rabi frequencies $\Omega_k$ of the $N$-level quantum chain producing perfect population transfer.\textsuperscript{12,13} One objective of this paper is to find all possible solutions. In this section, first we will solve for the time-dependent quantum dynamics and then generate all analytical solutions for perfect population transfer. The key step to solve for the quantum dynamics is to computing the exponential of the matrix $H$ (Eq. (8)).

First, we consider two-level and three-level chains, whose oscillations only have one Rabi frequency. The well-known Rabi oscillation for the two-level system (chain)\textsuperscript{27} is

$$C_2(t) = i \sin(\Omega_1 \tau_c(t)).$$

(14)

Thus, the solution of the control field with 100% yield is

$$\Omega_1 = (2k + 1) \frac{\pi}{2T_c}, \quad k = 0, 1, \ldots.$$  

(15)

The time-dependent quantum dynamics of the three-level chain is

$$C_3(t) = \frac{\Omega_1 \Omega_2}{\Omega_1^2 + \Omega_2^2} \left( \cos \left[ \sqrt{\Omega_1^2 + \Omega_2^2} \tau_c(t) \right] - 1 \right).$$

(16)

The control field can produce 100% yield only when

$$T_c \sqrt{\Omega_1^2 + \Omega_2^2} = (2k + 1)\pi,$$

(17a)

$$\Omega_1 = \Omega_2.$$  

(17b)

Thus, the solutions of the Rabi frequencies with 100% yield are

$$\Omega_1 = \Omega_2 = (2k + 1) \frac{\pi}{\sqrt{2} T_c}, \quad k = 0, 1, \ldots.$$  

(18)

The Rabi oscillations in the four-level and five-level chains have two Rabi frequencies. The time-dependent quantum dynamics of the four-level chain is

$$C_4(t) = -i \frac{\sqrt{2} \Omega_1 \Omega_2 \Omega_3}{\sqrt{a^2 - 4b}} \left( \frac{\sin \left[ \sqrt{a - \sqrt{a^2 - 4b} \tau_c(t)} \right]}{\sqrt{a - \sqrt{a^2 - 4b}}} - \frac{\sin \left[ \sqrt{a + \sqrt{a^2 - 4b} \tau_c(t)} \right]}{\sqrt{a + \sqrt{a^2 - 4b}}} \right),$$

(19)

with

$$a = \Omega_1^2 + \Omega_2^2 + \Omega_3^2,$$

(20a)

$$b = \Omega_1^2 \Omega_3^2.$$  

(20b)
The following inequalities may be derived

\[
|C_4(T_f)| \leq \frac{\sqrt{2\Omega_1\Omega_2\Omega_3}}{\sqrt{a^2-4b}} \left( \frac{1}{\sqrt{a-\sqrt{a^2-4b}}} + \frac{1}{\sqrt{a+\sqrt{a^2-4b}}} \right) \quad (21a)
\]

\[
= \frac{\Omega_2}{\sqrt{(\Omega_1-\Omega_3)^2+\Omega_2^2}} \quad (21b)
\]

\[
\leq 1. \quad (21c)
\]

The inequality (21a) becomes an equality only when

\[
\begin{aligned}
\sqrt{a-\sqrt{a^2-4b}} T_e &= (2k_1+1) \frac{\pi}{2}, \\
\sqrt{a+\sqrt{a^2-4b}} T_e &= (2k_1+1) \frac{\pi}{2} + (2k_2+1)\pi,
\end{aligned}
\]

(k_1, k_2 = 0, 1, \ldots). \quad (22)

The inequality (21c) becomes an equality only when

\[
\Omega_1 = \Omega_3. \quad (23)
\]

Hence, all the solutions for perfect population transfer are

\[
\begin{aligned}
\Omega_1 &= \Omega_3 = \frac{\pi}{T_e} \left( k_1 + \frac{1}{2} \right) \left( k_1 + 2k_2 + \frac{3}{2} \right), \\
\Omega_2 &= \frac{\pi}{T_e} (2k_2+1),
\end{aligned}
\]

(k_1, k_2 = 0, 1, \ldots). \quad (24)

The quantum dynamics of the five-level chain is

\[
C_5(t) = \frac{2\Omega_1\Omega_2\Omega_3\Omega_4}{\sqrt{d(c-\sqrt{d})}(c+\sqrt{d})} \left\{ -(c+\sqrt{d}) \cos \left[ \frac{\tau_c(t)}{\sqrt{2}} \sqrt{c-\sqrt{d}} \right] \\
+ (c-\sqrt{d}) \cos \left[ \frac{\tau_c(t)}{\sqrt{2}} \sqrt{c+\sqrt{d}} \right] + 2\sqrt{d} \right\} \quad (25)
\]

with

\[
c = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 + \Omega_4^2, \quad (26a)
\]

\[
d = (\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2)^2 + 4\Omega_2^2\Omega_3^2. \quad (26b)
\]

Similar to the case of the four-level chain, we may establish that the equality

\[
|C_5(T_f)| = 1 \quad (27)
\]

holds if and only if

\[
\begin{aligned}
\frac{T_e}{\sqrt{2}} \sqrt{c-\sqrt{d}} &= (2k_1+1)\pi, \\
\frac{T_e}{\sqrt{2}} \sqrt{c+\sqrt{d}} &= (2k_1+2k_2+3)\pi,
\end{aligned}
\]

(k_1, k_2 = 0, 1, \ldots). \quad (28)
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and

\[
\begin{cases}
\Omega_1 = \Omega_4 \\
\Omega_2 = \Omega_3
\end{cases}
\]  
(29)

Then, all the solutions for perfect transfer are

\[
\begin{cases}
\Omega_1 = \Omega_4 = \frac{\pi}{T_e} (k_1 + 1), \\
\Omega_2 = \Omega_3 = \frac{\pi}{T_e} \sqrt{2k_2 + 1}(2k_1 + 2k_2 + 3),
\end{cases}
\]
(30)

Now, consider the Rabi oscillations of a general \( N \)-level chain. The Hamiltonian \( H \) in Eq. (8) can be diagonalized as

\[HO = O\Delta,\]
(31)

where \( O \) is an orthogonal matrix whose columns are the eigenvectors of \( H \) and the diagonal matrix \( \Delta \) has the eigenvalues of \( H \) on the diagonal. The Rabi oscillations for the \( N \)-level chain can be expressed as

\[C_N = \sum_{j=1}^{N} O_1 O_{Nj} e^{i\Delta_j T_e}.\]
(32)

We readily see that the characteristic polynomials \( D_N(\Lambda) \) of Eq. (8) of various orders \( N \) satisfy a recursion relation

\[D_N(\Lambda) = \Lambda D_{N-1}(\Lambda) - \Omega_{N-1}^2 D_{N-2}(\Lambda).\]
(33)

The Hamiltonian characteristic polynomial of the \( N \)-level chain can be written as

\[
D_N(\Lambda) = \begin{cases}
\Lambda^N - (-1) \sum_{i=1}^{N-1} \Omega_i^2 \Lambda^{N-2} \\
+ \cdots + (-1)^i \sum_{k_{n+1} - k_n \geq 2} \Omega_{k_1}^2 \Omega_{k_2}^2 \cdots \Omega_{k_i}^2 \Lambda^{N-2i} \\
+ \cdots + (-1)^\frac{N}{2} \Omega_1^2 \Omega_3^2 \cdots \Omega_{N-1}^2,
\end{cases}
\]
for even \( N \)

\[
D_N(\Lambda) = \begin{cases}
\Lambda^N - (-1) \sum_{i=1}^{N-1} \Omega_i^2 \Lambda^{N-3} \\
+ \cdots + (-1)^i \sum_{k_{n+1} - k_n \geq 2} \Omega_{k_1}^2 \Omega_{k_2}^2 \cdots \Omega_{k_i}^2 \Lambda^{N-2i} \\
+ \cdots + (-1)^\frac{N-1}{2} \sum_{k_{n+1} - k_n \geq 2} \Omega_{k_1}^2 \Omega_{k_2}^2 \cdots \Omega_{k_{\frac{N-1}{2}}}^2,
\end{cases}
\]
for odd \( N \)

(34)
It is easy to see that the eigenvalue spectrum is symmetric about zero, and there are \([N/2]\) Rabi frequencies in the Rabi oscillations of the \(N\)-level chain.

4. Analytical Solutions for Perfect Population Transfer

In this section, we present another technique to obtain analytical solutions of Rabi frequencies for perfect population transfer of the pulse-driven quantum chains. All of the analytical solutions are obtained directly without solving for the dynamics of the systems.

4.1. Spectral conditions for perfect population transfer

The model for population transfer in this paper is similar to the one for information transfer,\(^{28}\) which was used to identify the conditions for perfect information transfer inside quantum spin-chains. It is easy to relate the results of the two models. Perfect population transfer depends on two conditions:

(i) Symmetry condition for Rabi frequencies \(\Omega_i\):

\[
\Omega_i = \Omega_{N-i}, \quad i = 1, 2, \ldots, N.
\]  

(ii) The eigenvalue spectrum can be expressed as

\[
\Lambda = \begin{cases} 
\text{diag} \left( -k_1, -k_2, \ldots, -k\left[\frac{N-1}{2}\right], k_0, k_1, \ldots, k\left[\frac{N-1}{2}\right] \right) \frac{\pi}{T_e}, & \text{for odd } N \\
\text{diag} \left( -k_0 - \frac{1}{2}, -k\left[\frac{N-1}{2}\right] - \frac{1}{2}, k_0 + \frac{1}{2}, \ldots, k\left[\frac{N-1}{2}\right] + \frac{1}{2} \right) \frac{\pi}{T_e}, & \text{for even } N,
\end{cases}
\]  

where \(\{k_i| i = 0, 1, \ldots, [(N-1)/2]\}\) is an ascending nonnegative integer sequence with the difference between neighbors being odd, and when \(N\) is odd, the element, \(k_0\), must be zero.

When the eigenvalue spectrum is determined, the search for the solutions \(\Omega_i\) becomes an inverse eigenvalue problem,\(^{29}\) which generally has to be solved numerically. In this paper, we use Gröbner basis analysis\(^{30}\) to solve this problem symbolically and obtain analytical solutions. It is assumed that the eigenvalues are sorted
in ascending order
\[ \Lambda_1 > \Lambda_2 > \cdots > \Lambda_{N-1} > \Lambda_N. \] (37)

Utilizing Vieta’s formulas\(^{31}\) and Eq. (34), we can get \([N/2]\) equations for the eigenvalues and the Rabi frequencies. When \(N\) is even, the equations take the form
\[
\begin{align*}
\sum^{N-1}_{i=1} \Omega_i^2 &= \sum^N_{j=1} \Lambda_j^2, \\
\cdots \\
\sum_{k_{n+1}-k_n \geq 2} \Omega_{k_1} \Omega_{k_2} \cdots \Omega_{k_i} &= \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq N} \Lambda_{j_1}^2 \Lambda_{j_2}^2 \cdots \Lambda_{j_i}^2, \\
\cdots \\
\Omega_1^2 \Omega_2^2 \cdots \Omega_{N-1}^2 &= \prod_{j=1}^N \Lambda_j^2. 
\end{align*}
\] (38)

When \(N\) is odd, there are \(N-1\) nonzero eigenvalues and the equations become
\[
\begin{align*}
\sum^{N-1}_{i=1} \Omega_i^2 &= \sum^{N-1}_{j=1} \Lambda_j^2, \\
\cdots \\
\sum_{k_{n+1}-k_n \geq 2} \Omega_{k_1} \Omega_{k_2} \cdots \Omega_{k_i} &= \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq N} \Lambda_{j_1}^2 \Lambda_{j_2}^2 \cdots \Lambda_{j_i}^2, \\
\cdots \\
\sum_{k_{n+1}-k_n \geq 2} \Omega_{k_1} \Omega_{k_2} \cdots \Omega_{k_i}^N &= \prod_{j=1}^{N-1} \Lambda_j^2. 
\end{align*}
\] (39)

### 4.2. Analytical solutions for the Rabi frequencies

The Gröbner bases analysis is used to solve the polynomial equations. A Gröbner basis is a particular type of generating subset of an ideal \(I\) in a polynomial ring \(R\). It provides a method to solve simultaneous polynomial equations. For example, there are actually three variables \((\Omega_1^2, \Omega_2^2, \Omega_3^2)\) in the case of a six-level chain from the symmetric condition Eq. (35). We can obtain the Gröbner bases from Eq. (38) with a certain monomial order. The first basis is a polynomial only involving \(\Omega_3^2\)
\[
[(\Lambda_1 + \Lambda_2 - \Lambda_3)^2 - \Omega_3^2][(\Lambda_1 - \Lambda_2 + \Lambda_3)^2 - \Omega_3^2] \\
\times [(-\Lambda_1 + \Lambda_2 + \Lambda_3)^2 - \Omega_3^2][(\Lambda_1 + \Lambda_2 + \Lambda_3)^2 - \Omega_3^2].
\] (40)
It is easy to see that there are four possible values for $\Omega_2^3$. The second and third Gröbner bases are too complex to be shown here, but they are linear functions of $\Omega_2^2$ and $\Omega_2^3$, respectively. Therefore, $\Omega_2^2$ and $\Omega_2^3$ can be uniquely determined by $\Omega_2^3$. Hence, there are four groups of solutions for $\Omega_2^k$, $k = 1, 2, 3$. After a careful check, we find that there is only one group fulfilling the condition that all variables are positive,

$$
\begin{align*}
\Omega_2^3 &= \Lambda_1 - \Lambda_2 + \Lambda_3 \\
\Omega_2^2 &= \Omega_4^2 = \frac{(\Lambda_1 - \Lambda_2)(\Lambda_2 - \Lambda_3)(\Lambda_1 + \Lambda_3)}{\Lambda_1 - \Lambda_2 + \Lambda_3} \\
\Omega_2^1 &= \Omega_5^2 = \Lambda_1 \Lambda_2 \Lambda_3
\end{align*}
$$

(41)

The Gröbner bases have the same structure for all chains with an even number of levels, and their analytical solutions can be obtained by the same technique. For instance, the analytical solutions for the four-level chains are

$$
\begin{align*}
\Omega_2 &= \Lambda_1 - \Lambda_2 \\
\Omega_2^2 &= \Lambda_1 \Lambda_2
\end{align*}
$$

(42)

which is equal to Eq. (24) in Sec. 3, and the analytical solutions for the eight-level chain are

$$
\begin{align*}
\Omega_4^{(8)} &= \Omega_3^{(6)} - \Lambda_4 \\
\Omega_9^{(8)} &= \Omega_5^{(8)} = \frac{\Omega_3^{(6)} \Omega_2^{(6)} + \Lambda_4 \Omega_3^{(6)}}{\Omega_4^{(8)}} \\
\Omega_2^{(8)} &= \Omega_6^{(8)} = \frac{\Omega_3^{(6)} \Omega_2^{(6)} \Omega_4^{(6)} - \Lambda_4 \Omega_3^{(6)}}{\Omega_4^{(8)}} \\
\Omega_4^{(8)} &= \Omega_7^{(8)} = \Lambda_4 \frac{\Omega_3^{(6)} \Omega_2^{(6)}}{\Omega_3^{(8)}}
\end{align*}
$$

(43)

which is a recursion relation with $\Omega_m^{(n)}$ being $m$th Rabi frequency of the $n$-level chain.

The Gröbner bases of the polynomial for chains with an odd number of levels have the same structure as for chains with an even number of levels but with different forms. For example, the first Gröbner bases of the seven-level chain is

$$
(\Lambda_1^2 + \Lambda_1^2 - \Lambda_3^2 - 2\Omega_3^2)(\Lambda_4^2 + \Lambda_1^2 - \Lambda_3^2 - 2\Omega_3^2)(\Lambda_1^2 - \Lambda_4^2 - \Lambda_3^2 + 2\Omega_3^2),
$$

(44)

which is a polynomial only in terms of $\Omega_3^2$. The second and third bases, which are similar to the cases of chains with an even even number of levels, are also linear.
functions of $\Omega_2^2$ and $\Omega_1^2$, respectively. After a similar analysis and choosing the group of solutions, which make all variables positive, we obtain the analytical solutions for the seven-level chain

$$
\begin{align*}
\Omega_7^2 &= \Omega_6^2 = \frac{1}{2}(\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2), \\
\Omega_5^2 &= \Omega_4^2 = \frac{(\Lambda_1^2 - \Lambda_2^2)(\Lambda_3^2 - \Lambda_4^2)}{\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2}, \\
\Omega_3^2 &= \Omega_2^2 = \frac{\Lambda_2^2\Lambda_4^2}{\Lambda_1^2 - \Lambda_2^2 + \Lambda_3^2}.
\end{align*}
$$

(45)

By this logic, we can obtain the analytical solutions of other chains with an odd number of levels. The analytical solutions for the three-level chain are

$$
\Omega_1^2 = \Omega_2^2 = \frac{\Lambda_1^2}{2}. \tag{46}
$$

The solutions for the five-level chain are

$$
\begin{align*}
\Omega_2^2 &= \Omega_3^2 = \frac{1}{2}(\Lambda_1^2 - \Lambda_2^2), \\
\Omega_1^2 &= \Omega_4^2 = \Lambda_2^2.
\end{align*} \tag{47}
$$

It is easy to see that the solutions for the Rabi frequencies of the three-level and five-level chains are the same as Eqs. (18) and (30) in Sec. 3. We can also get the solutions for the nine-level chain expressed as a recursion relation,

$$
\begin{align*}
\Omega_9^{(9)^2} &= \Omega_8^{(9)^2} = \Omega_7^{(9)^2} = \frac{\Lambda_1^2}{2}, \\
\Omega_6^{(9)^2} &= \Omega_5^{(9)^2} = \frac{\Omega_3^{(7)^2}\Omega_4^{(7)^2}}{\Omega_4^{(9)^2}} + \Lambda_4^2, \\
\Omega_4^{(9)^2} &= \Omega_3^{(9)^2} = \frac{\Omega_2^{(7)^2}\Omega_3^{(7)^2}}{\Omega_4^{(9)^2}\Omega_3^{(9)^2}} - \Lambda_4^2 \left( \frac{\Omega_2^{(7)^2}}{2\Omega_4^{(9)^2}} + \frac{\Omega_3^{(7)^2}}{\Omega_3^{(9)^2}} \right), \\
\Omega_1^{(9)^2} &= \Omega_8^{(9)^2} = \frac{\Omega_2^{(7)^2} + \Omega_4^{(7)^2}}{\Omega_3^{(9)^2}} - \Lambda_4^2.
\end{align*} \tag{48}
$$

where $\Omega_m^{(n)^2}$ denotes $m$th Rabi frequency of the $n$-level chain.

The analytical solutions for more than nine levels are too complex to display here, and numerical solutions would be a better choice. Moreover, the results show that there might be a recursion relation among the analytical solutions. This issue will be discussed in another paper.
4.3. The solutions with minimal fluence for the Rabi frequencies

Here, we consider the fluence for the Rabi frequencies, defined as their square summation

\[ F = \sum_{i=1}^{N-1} \Omega_i^2. \]  

(49)

From Eqs. (38) and (39), the following equality is valid for all chains:

\[ \sum_{i=1}^{N} \Omega_i^2 = \sum_{j=1}^{N} \Lambda_j^2. \]  

(50)

Hence, to get the optimal solution with the smallest fluence, we only need to minimize the fluence of the eigenvalues.

Since the eigenvalues \( \Lambda_j \) are nondegenerate, it is easy to see that following set of eigenvalues is optimal:

\[ \Lambda = \left\{ \begin{array}{ll} \left\{ -\left[ \frac{N}{2} \right], \ldots, \left[ \frac{N}{2} \right] \right\} T_e, & \text{for odd } N \\ \left\{ -\left[ \frac{N-1}{2} \right] - \frac{1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, \left[ \frac{N-1}{2} \right] + \frac{1}{2} \right\} T_e, & \text{for even } N \end{array} \right. \]  

(51)

We can further show that the optimal solutions can be expressed generally as

\[ \Omega_n = \sqrt{\frac{n(N-n)}{2T_e}} \pi, \quad n = 1, 2, \ldots, N-1. \]  

(52)

These solutions have been discussed previously,\(^{13}\) although their optimality was not considered.

5. Conclusions and Discussion

This paper explores the solutions for perfect population transfer in a pulse-driven finite-dimensional quantum chain with only nearest-neighbor coupling. The solutions are obtained from the time-dependent dynamics of the chain or from a Gröbner basis analysis of the polynomial equations that describe the connection between the Rabi frequencies and eigenvalues of the effective Hamiltonian. The concept of the Rabi oscillations is extended from two-level quantum chains to multi-level quantum chains. We explicitly obtained all solutions for the control pulses up to the nine-level chain. The solutions with minimal fluence for the Rabi frequencies are also found and expressed generally. The analytical solutions presented in this paper could have application to laboratory implementation of quantum control in suitable cases.

The recursion relationships amongst the analytical solutions for the Rabi frequencies and the recursion formula for the \( N \)-level chain will be in another work.\(^{32}\)
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Although the solutions in this paper only apply to population transfer from the ground state to the highest excited state in the quantum chains, the techniques can be extended to treat population transfer between any states in general multi-level quantum systems whose coupling is not only between neighboring levels.32

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